CCFU Proof 6

Lucas $= 2\cosh$, Fibonacci $= (2/\sqrt{5})\sinh$ (even index)

Given. Binet's formulas:

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \qquad L(n) = \varphi^n + \psi^n,$$

where $\psi = -1/\varphi$.

Even index. For $n = 2k$, $k \geq 0$:

$$\psi^{2k} = (-1/\varphi)^{2k} = (1/\varphi)^{2k} = \varphi^{-2k}.$$

Theorem 1 (Lucas $= 2\cosh$).

$$L(2k) = \varphi^{2k} + \varphi^{-2k} = 2\cosh(2k\ln\varphi). \qquad \blacksquare$$

Theorem 2 (Fibonacci $= (2/\sqrt{5})\sinh$).

$$F(2k) = \frac{\varphi^{2k} - \varphi^{-2k}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\sinh(2k\ln\varphi). \qquad \blacksquare$$

Corollary (Hyperbolic identity). Let $r = 2k\ln\varphi$. Then:

$$\begin{aligned} L(2k)^2 - 5\,F(2k)^2 &= [2\cosh(r)]^2 - 5\left[\frac{2}{\sqrt{5}}\sinh(r)\right]^2 \\ &= 4\cosh^2(r) - 4\sinh^2(r) \\ &= 4(\cosh^2(r) - \sinh^2(r)) \\ &= 4. \qquad \blacksquare \end{aligned}$$

Note. These identities hold exactly for even indices. For odd indices the roles swap: $L(2k+1)$ involves sinh and $F(2k+1)$ involves cosh. The even-index identities do not apply as stated to odd indices.